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Quantum effects in systems with accelerated mirrors

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Abstract. The problem of the vacuum state definition in a part of the Minkowski space bounded by a single or two spherically symmetric mirrors which expand or contract with uniform acceleration is considered. The Euclidean approach is used to obtain the explicit expressions for the corresponding causal Green functions. The vacuum stress–energy tensor is obtained and its properties are investigated.

1. Introduction

The problem of quantum particle creation by an external classical field is now of particular interest for many reasons. The case when an external field is of the special form of an infinite reflecting potential barrier whose position depends on time (the moving-mirror problem) seems to be the simplest one for investigation. At the same time, the consideration of the moving-mirror problem not only allows one to understand better more complicated situations (such as quantum particle creation by black holes) but also has its own importance.

It is well known that the vacuum state in the presence of even a static mirror will differ from the empty-space vacuum state because of the change in the boundary conditions. In the case when the mirror is moving, a new effect can occur—quantum particle creation (Moore 1970, DeWitt 1975, Fulling and Davies 1976, Davies and Fulling 1977, Candelas and Deutsch 1977). The conformal invariance of the massless-field equations allows one to reduce the problem in two-dimensional space–time with time-dependent boundaries to the problem with static boundaries which can be resolved explicitly. That is why almost all of the previous investigations have been devoted to the two-dimensional moving-mirror models, and the two-dimensional case is now understood well enough.

Although the two-dimensional results apparently allow one to understand qualitatively some of the properties of the quantum processes in real four-dimensional systems, they cannot be transferred to the four-dimensional case without some precautions. In particular, there are many well-known peculiar properties which distinguish the massless-field theory in two-dimensional space–time. The direct solution of the four-dimensional problem is usually much more complicated and, as far as we know, the only four-dimensional solution was found by Candelas and Deutsch (1977) for the case of a plane uniformly accelerated mirror.

In this paper a new method is proposed which allows us to find new solutions in an explicit closed form in the case when a single or pair of concentric spherical mirrors which expand or contract with uniform acceleration is present in four-dimensional space–time. After passing to imaginary time, the problem of the causal Green function construction in Minkowski space with boundaries reduces to the problem of the calculation of the point charge potential in four-dimensional Euclidean space in the presence of the spherical conducting boundary. The method of images can be applied to find a solution in this case. The causal Green function reconstructed by the analytical continuation to physical space–time of the Euclidean Green function obtained contains all necessary information about physical properties of the system, and in particular it allows us to find the vacuum stress–energy tensor. Using this method we obtain the corresponding Green function for interior and exterior regions of the spherical mirror which expands or contracts with uniform acceleration and for the region between two such concentric mirrors. The main feature of these problems is that the external field corresponding to the mirrors does not switch off in either the past or the future. This is why the quantum field cannot be considered as asymptotically free and the usual definition of in- and out-vacuum states cannot be applied. Nevertheless, the spaces of in- and out-particles states can be restored if only one has a special prescription which allows one to define the causal Green function (Menskii 1974, Rumpf 1976). Using the Green functions obtained by the above Euclidean approach, we prove that in the problems under consideration the corresponding vacuums are stable, and calculate the stress–energy tensor describing the vacuum polarisation.

The following notations are used in our Paper. The Minkowski metric signature is $(-+++)$, and the scalar product of two vectors x and y with respect to this metric is denoted xy. The inner product of vectors x and y in a Euclidean space with a signature $(++++)$ is denoted (xy), and

$$|x|^2=\sum_{i=1}^{4}(x_i)^2.$$

The symbols $x > y$, $x > \Sigma$ ($x < y$, $x < \Sigma$) mean that a point X lies in the future (in the past) of a point y and a surface Σ respectively. A complex conjugate is denoted by a bar over a symbol.

2. Euclidean theory, Green functions and stress–energy tensor

We restrict ourselves to considering the theory of the massless scalar field $\phi(x)$ with an action

$$S[\phi]=-\tfrac{1}{2}\int(-g)(\phi,_{\mu}\phi^{,\mu}+\xi R\phi^2)\,\mathrm{d}x. \tag{2.1}$$

For conformal invariant theory $\xi=\frac{1}{6}$. The variation of this action with respect to $g_{\mu\nu}$ gives the stress–energy operator. In flat space–time it is of the form

$$T^{\mu\nu}_{(\xi)}(x)=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}=\lim_{y\to x}D^{\mu\nu}_{(\xi)}(x,y)[\phi(x),\phi(y)]_+, \tag{2.2a}$$

where $[A,B]_+=AB+BA$ and

$$D^{\mu\nu}_{(\xi)}(x,y)=(\tfrac{1}{2}-\xi)\nabla^{\mu}_{x}\nabla^{\nu}_{y}-(\tfrac{1}{4}-\xi)g^{\mu\nu}g_{\alpha\beta}\nabla^{\alpha}_{x}\nabla^{\beta}_{y}-\tfrac{1}{2}\xi(\nabla^{\mu}_{x}\nabla^{\nu}_{x}\nabla^{\mu}_{y}\nabla^{\nu}_{y}). \tag{2.2b}$$

In Cartesian coordinates one can put $\nabla^{\mu}_{x}=\partial/\partial x_{\mu}$.

The field operator $\boldsymbol{\phi}$ in the Heisenberg representation satisfies the equation

$$\Box\boldsymbol{\phi}=0, \tag{2.3a}$$

with the following boundary condition on the moving-mirror surface Σ:

$$\boldsymbol{\phi}|_{\Sigma}=0. \tag{2.3b}$$

Denote by $|0;\text{in}\rangle$ ($|0;\text{out}\rangle$) the in-vacuum (out-vacuum) state for the problem under consideration. The concrete choice of these states will be discussed later. The problem of the vacuum stress–energy tensor definition in the region outside the mirror's boundary can be reduced to the computation of the matrix elements

$$T^{\mu\nu}_{(\xi)}(x)=\langle 0;\text{in}|\boldsymbol{T}^{\mu\nu}_{(\xi)}(x)|0;\text{in}\rangle=\lim_{y\to x}D^{\mu\nu}_{(\xi)}(x,y)G^{(1)}(x|y). \tag{2.4}$$

Here $G^{(1)}(x|y)$ is a Hadamard function (a fundamental solution)

$$G^{(1)}(x|y)=\langle 0;\text{in}|[\boldsymbol{\phi}(x),\boldsymbol{\phi}(y)]_{+}|0;\text{out}\rangle. \tag{2.5}$$

$G^{(1)}(x|y)$ is a symmetric function of x and y and satisfies equation (2.3a) and boundary conditions

$$G^{(1)}(x|y)|_{x\in\Sigma}=G^{(1)}(x|y)|_{y\in\Sigma}=0.$$

When the product of the field operators at coinciding points in equations (2.4) and (2.5) is taken, the corresponding expectation value is evidently divergent. In the case under consideration, to remove these divergences it is sufficient to subtract the corresponding zero-point vacuum fluctuations in the Minkowski space without boundaries. The regularised stress–energy tensor is

$$T^{\mu\nu}_{(\xi)\text{reg}}(x)=\lim_{y\to x}D^{\mu\nu}_{(\xi)}(x,y)G^{(1)}_{\text{reg}}(x|y). \tag{2.6}$$

The subscript 'reg' for any Green function denotes that the difference between this Green function and the corresponding 'free' (i.e. in Minkowski space without boundaries) Green function (labelled by subscript 'o') is taken. For example,

$$G^{(1)}_{\text{reg}}(x|y)=G^{(1)}(x|y)-G^{(1)}_{0}(x|y). \tag{2.7}$$

It is also convenient to introduce the function

$$G^{(\text{in})}(x|y)=\mathrm{i}\langle 0;\text{in}|T(\boldsymbol{\phi}(x)\boldsymbol{\phi}(y))|0;\text{in}\rangle, \tag{2.8}$$

connected with the Hadamard function $G^{(1)}(x|y)$ by

$$G^{(\text{in})}(x|y)=\tilde{G}(x|y)+\tfrac{1}{2}\mathrm{i}G^{(1)}(x|y), \tag{2.9}$$

where

$$\begin{aligned}&\tilde{G}(x|y)=\mathrm{i}\epsilon(x^0-y^0)[\boldsymbol{\phi}(x),\boldsymbol{\phi}(y)],\\&\epsilon(\alpha)\equiv\tfrac{1}{2}(\theta(\alpha)-\theta(-\alpha)),\\&[\boldsymbol{A},\boldsymbol{B}]\equiv\boldsymbol{AB}-\boldsymbol{BA}.\end{aligned} \tag{2.10}$$

Using the canonical commutation relations it is not difficult to verify that when the points x and y lie on some space-like surface both the functions $\tilde{G}(x|y)$, $\tilde{G}_0(x|y)$ and their derivatives coincide. Thus one has

$$T^{\mu\nu}_{(\xi)\text{reg}}(x)=-2\mathrm{i}\lim_{y\to x}D^{\mu\nu}_{(\xi)}(x,y)G^{(\text{in})}_{\text{reg}}(x|y). \tag{2.11}$$

From now on we assume that the limit $y \to x$ is taken along a space-like direction. Introduce also a causal (Feynman) Green function defined by the relation

$$G(x|y) = \mathrm{i}\langle 0;\mathrm{out}|T(\boldsymbol{\phi}(x)\boldsymbol{\phi}(y))|0;\mathrm{in}\rangle/\langle 0;\mathrm{out}|0;\mathrm{in}\rangle. \tag{2.12}$$

This Green function contains all necessary information about the considered quantum system in an external field. Using it, one can in principle reconstruct the spaces of in- and out-particles states and define the S matrix (see e.g. Menskii 1974, Rumpf 1976 and appendix 1 of this paper).

In particular, it can be verified (see appendix 1) that the vacuum stability condition $|0;\mathrm{out}\rangle = \mathrm{e}^{\mathrm{i}\alpha}|0;\mathrm{in}\rangle$ is equivalent to the fulfilment of the relation

$$(G * \bar{G})_\Sigma(x|z) = 0, \qquad x > \Sigma > z, \tag{2.13}$$

where

$$(G * \bar{G})_\Sigma(x|z) = \int_\Sigma \left(G(x|y)\frac{\overleftrightarrow{\partial}}{\partial y^\mu}\bar{G}(y|z)\right)\mathrm{d}\Sigma^\mu(y)$$

and Σ is some complete Cauchy surface. If the vacuum state is stable, then $G(x|y) = G^{(\mathrm{in})}(x|y)$, and using the causal Green function one can easily calculate $T^{\mu\nu}_{(\xi)\,\mathrm{reg}}(x)$ defined by equation (2.11).

In the more general case when particle creation is possible, the connection between $G(x|y)$ and $G^{(\mathrm{in})}(x|y)$ becomes more complicated. It can be shown (see appendix 1) that $G^{(\mathrm{in})}(x|y)$ can be defined as a solution of the following system of equations:

$$\begin{aligned} &(G * G^{(\mathrm{in})})_\Sigma(x|z) = G(x|z), \qquad x > \Sigma > z,\\ &(\bar{G} * G^{(\mathrm{in})})_\Sigma(x|z) = 0, \qquad x > \Sigma > z,\\ &G^{(\mathrm{in})}(x|y) = G^{(\mathrm{in})}(y|x). \end{aligned} \tag{2.14}$$

If the field sources switch off in the remote past and future, then for a fixed value y the function $G(x|y)$ defined by equation (2.12) contains only positive (negative) frequencies when $x^0 \to \infty$ ($x^0 \to -\infty$). After the Wick rotation $x^0 \to -\mathrm{i}x_4$, the corresponding Euclidean Green function $\mathscr{G}(x_4, \boldsymbol{x}|y_4, \boldsymbol{y}) = -\mathrm{i}G(-\mathrm{i}x_4, \boldsymbol{x}|-\mathrm{i}y_4, \boldsymbol{y})$ decreases at the Euclidean space infinity. (The discussion of the Euclidean formulation of the quantum field theory can be found, for example, in Schwinger (1970).) In the Minkowski space without boundary, the causal Green function $G(x|y)$ can be defined as a special analytic continuation of a Euclidean Green function $\mathscr{G}(x|y)$, which is uniquely defined as a decreasing-at-infinity solution of the equation

$$\mathscr{L}(x)\mathscr{G}(x|y) = \mathscr{L}(y)\mathscr{G}(x|y) = -\delta(x-y),$$

where

$$\mathscr{L}(x) \equiv \sum_{i=1}^{4} \frac{\partial^2}{(\partial x_i)^2}.$$

In a more general case (in a curved space–time or in the presence of a non-vanishing external field or boundaries in a Minkowski space), when definition (2.12) cannot be applied, it appears that sometimes the corresponding Euclidean Green function can be naturally defined (see e.g. Hartle and Hawking 1976, Chitre and Hartle 1977, Lapedes 1978). In this case it is natural to try to use the Euclidean Green function analytic continuation to define unambiguously the corresponding causal Green function $G(x|y)$ in original physical space–time. In our paper we use this approach.

3. Geometry of models and method of images

3.1. Problem A

As a first example we consider the case when the mirror boundary surface Σ is described by the equation

$$x_\mu x^\mu = \boldsymbol{x}^2 - t^2 = a^2. \tag{3.1}$$

The points of this spherical mirror are moving with a constant (in their own reference system) acceleration a^{-1}. A simple geometrical consideration shows that a null ray can meet the reflecting surface of the mirror not more than once. (In the case of the uniformly accelerated plane considered by Candelas and Deutsch (1977), almost all null rays are reflected by this plane an infinite number of times.)

The Euclidean section Σ_E ($t = -\mathrm{i}\tau$) of an analytic continuation of the boundary surface Σ is a four-dimensional sphere S^4 (figure 1),

$$\boldsymbol{x}^2 + \tau^2 = a^2. \tag{3.2}$$

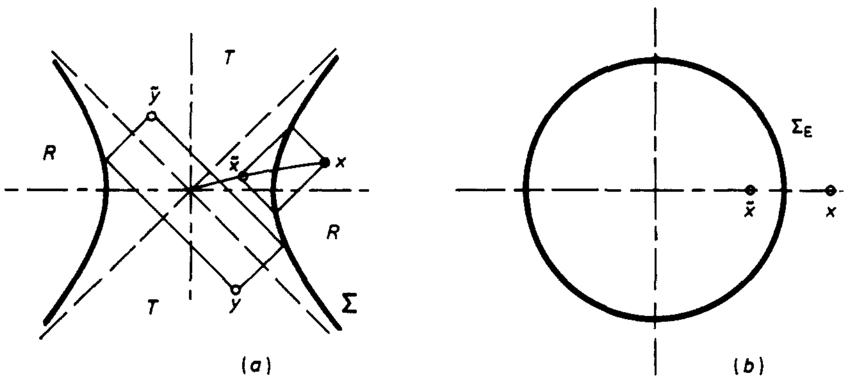

Figure 1. The geometry of problem A. The bold curves show (a) the uniformly accelerated spherical mirror and (b) the corresponding Euclidean section. The points x, y, ... and their images $\tilde{x}$, $\tilde{y}$, ... are shown both in Minkowski and in Euclidean space. All the null rays emitted from y will pass after reflection through the point $\tilde{y}$.

Let B^4 denote the interior part of this sphere. The corresponding Euclidean Green function $\mathscr{G}(x|y)$ satisfies the equation

$$\mathscr{L}(x)\mathscr{G}(x|y) = -\delta(x-y) \tag{3.3a}$$

and the boundary condition

$$\mathscr{G}(x|y)|_{x\in\Sigma_\mathrm{E}} = \mathscr{G}(x|y)|_{y\in\Sigma_\mathrm{E}} = 0. \tag{3.3b}$$

(If x, $y \bar{\in} B^4$ (an external problem), then the function $\mathscr{G}(x|y)$ is considered to be decreasing when one of its arguments tends to infinity.) Thus the function $\mathscr{G}(x|y)$ can be considered as a potential, at a point x, of a unit point charge located at a point y in four-dimensional space in the presence of a conducting sphere. This problem can be resolved by the method of images in the same manner as for the analogous problem in electrostatics.

Let

$$\tilde{x} = J_a x \equiv (a^2/|x|^2)x \tag{3.4}$$

be an image of the point x under an inversion transformation with respect to a sphere of a radius a, the centre of which is located at the origin of the coordinate system ($|x|^2 = a^2$). Then the solution of equations (3.3) can be given in the form

$$\begin{aligned}\mathscr{E}(x|y) &= (\boldsymbol{I} + \boldsymbol{J}_a(x))\mathscr{E}_0(x|y)\\ &= (\boldsymbol{I} + \boldsymbol{J}_a(y))\mathscr{E}_0(x|y)\\ &= \frac{1}{4\pi^2}\left(\frac{1}{|x-y|^2} - \frac{a^2/|y|^2}{|x-(a^2/|y|^2)y|^2}\right),\end{aligned} \tag{3.5}$$

where $\boldsymbol{J}_a(x)f(x) \equiv -(a^2/|x|^2)f(\tilde{x})$, and $\boldsymbol{I}$ is an identity operator. Since the function $\mathscr{E}(x|y)$ is symmetric (i.e. $\mathscr{E}(x|y) = \mathscr{E}(y|x)$) and equations (3.3) are invariant under the inversion transformation, equation (3.5) for $x, y \in B^4 (x, y \bar{\in} B^4)$ gives the interior (exterior) solution of problem (3.3): The analytic continuation of this function allows one to define the causal Green function both inside and outside the accelerated spherical mirror.

3.2. Problem B

The same method can be exploited to solve the corresponding problem in the region between two concentric mirror surfaces Σ^1 and Σ^2 given by

$$\begin{aligned}&\Sigma^1\colon \boldsymbol{x}^2 - t^2 = a^2,\\ &\Sigma^2\colon \boldsymbol{x}^2 - t^2 = b^2, \qquad b > a.\end{aligned} \tag{3.6}$$

The Euclidean section of the analytic continuation of this region is a spherical layer between two four-dimensional spheres (figure 2):

$$\begin{aligned}&\Sigma^1_{\mathrm{E}}\colon \boldsymbol{x}^2 + \tau^2 = a^2,\\ &\Sigma^2_{\mathrm{E}}\colon \boldsymbol{x}^2 + \tau^2 = b^2, \qquad b > a.\end{aligned} \tag{3.7}$$

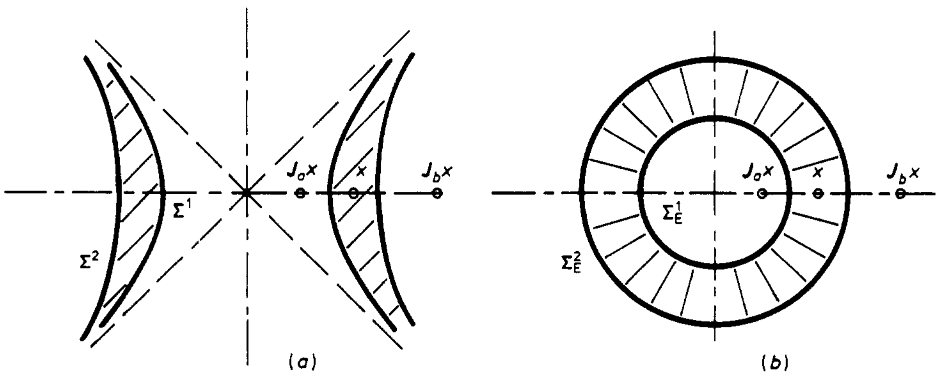

Figure 2. The geometry of problem B. The bold curves show (*a*) the uniformly accelerated concentric spherical mirrors and (*b*) the corresponding Euclidean sections. The images of point x are shown both in Minkowski and in Euclidean space.

In this case the method of images can be used to find the solution of equation (3.3) in this region $B_{\rm E}(\partial B_{\rm E}=\Sigma_{\rm E}=\Sigma_{\rm E}^1\cup\Sigma_{\rm E}^2)$ in the form

$$\mathscr{E}(x|y)=(\boldsymbol{I}+\boldsymbol{K}(x))\mathscr{E}_0(x|y)=(\boldsymbol{I}+\boldsymbol{K}(y))\mathscr{E}_0(x|y), \tag{3.8}$$

where

$$\begin{aligned}\boldsymbol{K}&=\boldsymbol{J}_a+\boldsymbol{J}_A\boldsymbol{J}_b+\boldsymbol{J}_a\boldsymbol{J}_b\boldsymbol{J}_a+\ldots+\boldsymbol{J}_b+\boldsymbol{J}_b\boldsymbol{J}_a+\boldsymbol{J}_b\boldsymbol{J}_a\boldsymbol{J}_b+\ldots\\&=(\boldsymbol{L}-\boldsymbol{J}_a\boldsymbol{J}_b)^{-1}(\boldsymbol{J}_a+\boldsymbol{J}_a\boldsymbol{J}_b)+(\boldsymbol{L}-\boldsymbol{J}_b\boldsymbol{J}_a)^{-1}(\boldsymbol{J}_b+\boldsymbol{J}_b\boldsymbol{J}_a).\end{aligned} \tag{3.9}$$

The explicit expression for $\mathscr{E}(x|y)$ can be written as

$$\mathscr{E}(x|y)=\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\left[\left(\frac{a}{b}\right)^{2n}\left(\frac{1}{|x-(a/b)^{2n}y|^2}-\frac{b^2}{|x|^2|y|^2-2b^2(a/b)^{2n}(xy)+(a/b)^{4n}b^4}\right)\right]. \tag{3.10}$$

The analytic continuation of solutions (3.5) and (3.10) to physical space–time will be discussed in the next section.

It should be emphasised that the method of images in Euclidean space can be exploited to define the causal Green function for a wide class of moving-mirror boundaries. Let the Euclidean boundary consist of the parts of spheres and planes (a boundary may be connected or not, as in example B). If there exists a (finite or infinite) number of symmetry transformations (reflections and inversions) which allows one to map a considered region onto the total Euclidean space, leaving the boundary of the region unchanged, then one can obtain the solution to equation (3.3) in the form (3.8), where $\boldsymbol{K}$ is an operator constructed from the operators of the reflections and inversions by certain rules.

4. Causal Green function, vacuum stability and stress–energy tensor

Consider now $\mathscr{E}_0(x|y)=1/4\pi^2|x-y|^2$ as an analytic function of two complex vector variables x and y. This function is well-defined everywhere except at the points where the denominator becomes zero. If x and y are real, then for $x_4=\mathrm{i}x^0$ and $y_4=\mathrm{i}y^0$ the analytical continuation of $\mathscr{E}_0(x|y)$ is regular everywhere except at the points where $(x-y)^2=0$. The usual way to define this function at these points as a distribution is to use the Wick rotation prescription

$$G_0(x|y)=\mathrm{i}\mathscr{E}_0(x_4=\mathrm{i}(1-\mathrm{i}\epsilon)x^0,\boldsymbol{x}|y_4=\mathrm{i}(1-\mathrm{i}\epsilon)y^0,\boldsymbol{y})=\frac{\mathrm{i}}{4\pi^2}\frac{1}{(x-y)^2+\mathrm{i}\epsilon}. \tag{4.1}$$

We postulate that the same infinitesimal term $\mathrm{i}\epsilon$ should be added to the denominators of all analytically continued parts of the Euclidean function under consideration to define the causal Green function as a distribution in physical space–time correctly. The corresponding expressions for the causal Green functions for problems A and B considered in the previous section are given in table 1.

It should be noted that if a point x lies in the R region $(x^2>0)$, then $\tilde{x}=\boldsymbol{J}_ax$ also lies in the R region and the points x and $\tilde{x}$ are separated by the mirror surface. In the case of the interior problem A, when $x^2<0$ the image $\tilde{x}=\boldsymbol{J}_ax$ lies inside the mirror and we have

$$\Box G(x|y)=\delta(x-y)-(a^2/y^2)\delta(x-(a^2/y^2)y).$$

Table 1

Problem	Causal Green function	Canonical stress–energy tensor $\langle 0\lvert \boldsymbol{T}_{(\xi=0)\mu\nu}\rvert 0\rangle$	Improved stress–energy tensor $\langle 0\lvert \boldsymbol{T}_{(\xi=1/6)\mu\nu}\rvert 0\rangle$
A. *Single mirror*	$\frac{\mathrm{i}}{4\pi^2}\left(\frac{1}{(x-y)^2+\mathrm{i}\epsilon} - \frac{a^2}{x^2y^2-2a^2xy+a^4+\mathrm{i}\epsilon}\right)$	$-\frac{a^2}{\pi^2}(a^2-x^2)^{-4} \cdot [x_\mu x_\nu - \frac{1}{2}\eta_{\mu\nu}(a^2+x^2)]$	0
B. *Pair of concentric mirrors*	$\frac{\mathrm{i}}{4\pi^2}\sum_{n=-\infty}^{\infty}\left[\left(\frac{a}{b}\right)^n \cdot \left(\frac{1}{(x-(a/b)^{2n}y)+\mathrm{i}\epsilon} - \frac{b^2}{x^2y^2-2b^2(a/b)^{2n}xy+(a/b)^{4n}b^4+\mathrm{i}\epsilon}\right)\right]$	$\frac{1}{\pi^2}\sum_{n=-\infty}^{\infty}{}' \left\{2(x_\mu x_\nu - \frac{1}{4}\eta_{\mu\nu}x^2) \cdot (x^2)^{-3}\left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right]^{-4} - b^2\left(\frac{a}{b}\right)^{2n}\left[x_\mu x_\nu - \frac{1}{2}\eta_{\mu\nu}\left(b^2\left(\frac{a}{b}\right)^{2n} + x^2\right)\right] \cdot \left[b^2\left(\frac{a}{b}\right)^{2n} - x^2\right]^{-4}\right\}$	$-\frac{2}{3\pi^2}(x_\mu x_\nu - \frac{1}{4}\eta_{\mu\nu}x^2) \cdot (x^2)^{-3}\sum_{h=-\infty}^{\infty}{}' \left\{\left[2+\left(\frac{b}{a}\right)^{2n}\right] \cdot \left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right]^{-4}\right\}$

The appearance of the second δ function term on the right-hand side of this equation can be understood if we note that, because of the geometry of our model, all the null rays being emitted from a point x inside the uniformly accelerated mirror will reach the point $\tilde{x} = J_a x$ (figure 1). The factor $-a^2/y^2$ appears because of the Doppler change of the intensity of the light reflected by the moving mirror. For the same reason the causal Green function in the Einstein world possesses the same property.

Using the above-defined causal Green functions, one can verify (see appendix 2) that the corresponding vacuum states inside and outside the spherical mirror which expands or contracts with uniform acceleration (problem A) and in the region between two such concentric mirrors (problem B) are stable. Thus

$$T^{\mu\nu}_{(\xi)\,\mathrm{reg}}(x) = -2\mathrm{i} \lim_{y\to x} D^{\mu\nu}_{(\xi)}(x, y) G_{\mathrm{reg}}(x|y). \tag{4.2}$$

It should be noted that outside the mirror boundary $G_{\mathrm{reg}}(x|y)$ is a regular function for $x = y$, and the iϵ addition is essential only when the stress–energy tensor behaviour near the mirror boundary is analysed. The explicit expressions for the canonical ($\phi = 0$) and improved ($\xi = \frac{1}{6}$) stress–energy tensors in the problems under consideration are given in table 1. We discuss now the properties of the vacuum stress–energy tensors obtained.

4.1. *Single spherical accelerated mirror* (*A*)

We begin by considering the canonical ($\xi = 0$) stress–energy tensor properties. The corresponding energy density distribution at time $t = 0$ is

$$T_{(\xi=0)00} = -\frac{a^2}{2\pi^2}\frac{\boldsymbol{x}^2 + a^2}{(\boldsymbol{x}^2 - a^2)^4}, \tag{4.3}$$

i.e. the energy density is everywhere (both inside and outside) negative. Far from the mirror when $|\boldsymbol{x}| \to \infty$ one obtains

$$T_{(\xi=0)00} \sim -\frac{a^2}{2\pi^2}\frac{1}{|\boldsymbol{x}|^6}. \tag{4.4}$$

Near the mirror boundary the negative energy density increases infinitely:

$$T_{(\xi=0)00} \sim -\frac{1}{16\pi^2}\frac{1}{\Delta^4}, \qquad \Delta \equiv |\boldsymbol{x}| - a. \tag{4.5}$$

This behaviour is also characteristic of the canonical vacuum energy density near the conducting plane at rest. This divergence is well known to be connected with the non-physical choice of the boundary conditions. Namely, the mirror is considered to be ideal, i.e. (a) it reflects the waves of all the frequencies completely, and (b) the mirror reaction on any external field change is supposed to be instantaneous.

An asymptotic behaviour of the stress–energy tensor when a point p tends to the future null infinity ($p \to \mathscr{I}^+$) along the null geodesic $x^\mu = ut^\mu + rl^\mu(\theta, \phi)(t_\mu t^\mu = -1$, $l_\mu t^\mu = -1$, $l_\mu l^\mu = 0)$ is

$$r^2 T_{(\xi=0)\mu\nu} \sim -(a^2/16\pi^2 u^4) l_\mu l_\nu \qquad (r \to \infty). \tag{4.6}$$

This asymptotic form shows that there is a negative energy density flux from the mirror at infinity. The analogous asymptotic behaviour near $\mathscr{I}^-$(a point p tends to $\mathscr{I}^-$ along a null geodesic $x^\mu = vt^\mu + rn^\mu(\theta, \phi)$, $n_\mu n^\mu = 0$, $n_\mu t^\mu = 1$) is

$$r^2 T_{(\xi=0)\mu\nu} \sim -(a^2/16\pi^2 v^4) n_\mu n_\nu. \tag{4.7}$$

It describes the flux of the negative energy incident on the mirror from the $\mathscr{I}^-$ infinity.

When $a \to \infty$, we have a mirror plane at rest, and the corresponding stress–energy tensor limit

$$T_{(\xi=0)\mu\nu} = \frac{1}{16\pi^2} \frac{\eta_{\mu\nu} - \delta^1_\mu \delta^1_\nu}{(x^1)^4} \tag{4.8}$$

coincides with the well-known expression for the vacuum canonical stress–energy tensor in the half-space over the infinite conducting plane (see e.g. DeWitt 1975). Rectangular coordinates are chosen in such a way that $x^1 = 0$ is an equation of the plane mirror surface.

The improved stress–energy tensor is identically zero for the problem A.

4.2. *Space–time region between two concentric spherical accelerated mirrors* (*B*)

In this case both canonical and improved vacuum stress–energy tensors do not vanish. The improved energy density at time $t = 0$ is

$$T_{(\xi=\frac{1}{6})00} = -\frac{1}{6\pi^2} \frac{1}{|\boldsymbol{x}|^4} \sum_{n=-\infty}^{\infty}{}' \frac{2 + (b/a)^{2n}}{[(b/a)^n - (a/b)^n]^4}. \tag{4.9}$$

(The prime denotes that the term with $n = 0$ in a sum must be omitted.) If the limit $a \to \infty$, $b \to \infty$, $d = b - a =$ constant is taken, one obtains two parallel mirror plates, and the corresponding vacuum stress–energy limit is

$$T_{(\xi=0)\mu\nu} = \frac{\pi^2}{1440 d^4} (\eta_{\mu\nu} - 4\delta^1_\mu \delta^1_\nu) + \frac{1}{16\pi^2 d^4} (\eta_{\mu\nu} - \delta^1_\mu \delta^1_\nu) \sum_{n=-\infty}^{\infty}{}' \left(n + \frac{x^1}{d}\right)^{-4}, \tag{4.10}$$

$$T_{(\xi=\frac{1}{6})\mu\nu} = \frac{\pi^2}{1440 d^4} (\eta_{\mu\nu} - 4\delta^1_\mu \delta^1_\nu). \tag{4.11}$$

It is not difficult to show that these expressions coincide with the well-known results obtained earlier for this case (see DeWitt 1975).

If we compare the results obtained with the results for the plane accelerated mirror we can conclude that many properties of the two-dimensional models are valid in our case and not in the plane accelerated mirror case. For example, the improved stress–energy tensor is equal to zero in problem A. It is rather remarkable that in the conformally coupled case there is no energy either inside or outside the sphere, even though when the sphere is static it is well known that a Casimir energy exists. This Casimir energy is apparently exactly cancelled by the polarisation energy produced by the surface of an expanding or contracting mirror when it is uniformly accelerated.

It should be stressed that the method developed in this paper can apparently be exploited when other (i.e. Dirac, Maxwell, etc) massless fields and more complicated boundaries are considered.

Appendix 1. Causal Green function and the vacuum stability condition

Let $\mathscr{F}$ be a space of complex solutions of the linear equation $D[f] = 0$, and $B(f_1, f_2)$ be a canonical bilinear form corresponding to this equation calculated for two complex solutions $f_1, f_2 \in \mathscr{F}$. For the theory of the scalar field described by the action (2.1),

$$B(f_1, f_2) = \int_\Sigma (f_1 \partial_\mu f_2 - f_2 \partial_\mu f_1)\, \mathrm{d}\Sigma^\mu.$$

Suppose that two normalised bases $F_{\text{in}} = (f_{\text{in},\alpha}, \bar{f}_{\text{in},\alpha})$ and $F_{\text{out}} = (f_{\text{out},\alpha}, \bar{f}_{\text{out},\alpha})$ are given in $\mathscr{F}$ which correspond to some particular choice of the decomposition of the positive and negative frequencies in the past (F_{in}) and in the future (F_{out}). In this case the in- (out-) vacuum state can be defined by conditions

$$\boldsymbol{a}_{\text{in},\alpha}|0;\text{in}\rangle = \mathrm{i}B(\bar{f}_{\text{in},\alpha}\boldsymbol{\phi})|0;\text{in}\rangle = 0,$$

$$(\boldsymbol{a}_{\text{out},\alpha}|0;\text{out}\rangle = \mathrm{i}B(\bar{f}_{\text{out},\alpha}\boldsymbol{\phi})|0;\text{out}\rangle = 0).$$

To obtain the expression for the causal Green function $G(x|y)$ it is convenient to use the formula

$$\boldsymbol{\phi}(x) = \sum_{\alpha,\beta} (\overset{+}{\boldsymbol{a}}_{\text{out},\alpha} A^{-1}_{\alpha\beta} \bar{f}_{\text{in},\alpha} + f_{\text{out},\alpha} A^{-1}_{\alpha\beta} \boldsymbol{a}_{\text{in},\beta}), \tag{A1.1}$$

where

$$A_{\alpha\beta} \equiv \mathrm{i}B(\bar{f}_{\text{in},\alpha}, f_{\text{out},\beta}) = [\boldsymbol{a}_{\text{in},\alpha}, \overset{+}{\boldsymbol{a}}_{\text{out},\beta}]. \tag{A1.2}$$

A substitution of (A1.1) into the definition (2.12) gives ($S_0 = \langle 0;\text{out}|0;\text{in}\rangle$)

$$G(x|y) = \mathrm{i}S_0^{-1} \begin{cases} \sum_{\alpha,\beta} f_{\text{out},\alpha}(x) A^{-1}_{\alpha\beta} \bar{f}_{\text{in},\beta}(y), & x > y \\ \sum_{\alpha,\beta} f_{\text{out},\alpha}(y) A^{-1}_{\alpha\beta} \bar{f}_{\text{in},\beta}(x), & x < y. \end{cases} \tag{A1.3}$$

For any functions $G(y_1, \ldots, y_n|x)$ and $F(x|z_1, \ldots, z_n)$ considered for fixed values of $y_1, y_2, \ldots, y_n$ and $z_1, z_2, \ldots, z_m$ we introduce the notation

$${}_{y_1,y_2\ldots y_n}G(x) \equiv G(y_1, y_2, \ldots, y_n|x),$$

$$F_{z_1 z_2 \ldots z_m}(x) \equiv F(x|z_1, z_2, \ldots, z_m),$$

and denote

$$(G * F)_\Sigma(y_1, \ldots, y_n|z_1, \ldots, z_m) = B_\Sigma({}_{y_1\ldots y_n}G, F_{z_1\ldots z_m}). \tag{A1.4}$$

Using equation (A1.3) it is not difficult to show that, for any function f, $(G * f)_\Sigma(x)$ $((\bar{G} * f)_\Sigma(x))$ contains only positive (negative) frequencies in the future (for $x > \Sigma$) and contains only negative (positive) frequencies in the past (for $x < \Sigma$).

One can also rewrite the expression for the Green function $G(x|y)$ in two equivalent forms:

$$G(x|y) = \begin{cases} \mathrm{i}\Big(\sum_\alpha f_{\text{out},\alpha}(x) \bar{f}_{\text{out},\alpha}(y) + \sum_{\alpha,\beta} f_{\text{out},\alpha}(x) V_{\alpha\beta} f_{\text{out},\beta}(y)\Big), & x > y \\ \mathrm{i}\Big(\sum_\alpha f_{\text{out},\alpha}(y) \bar{f}_{\text{out},\alpha}(x) + \sum_{\alpha,\beta} f_{\text{out},\alpha}(y) V_{\alpha\beta} f_{\text{out},\beta}(x)\Big), & x < y, \end{cases} \tag{A1.5a}$$

$$G(x|y) = \begin{cases} \mathrm{i}\Big(\sum_\alpha f_{\text{in},\alpha}(x) \bar{f}_{\text{in},\alpha}(y) + \sum_{\alpha,\beta} \bar{f}_{\text{in},\alpha}(x) \Lambda_{\alpha\beta} \bar{f}_{\text{in},\beta}(y)\Big), & x > y \\ \mathrm{i}\Big(\sum_\alpha f_{\text{in},\alpha}(y) \bar{f}_{\text{in},\alpha}(x) + \sum_{\alpha,\beta} \bar{f}_{\text{in},\alpha}(y) \Lambda_{\alpha\beta} \bar{f}_{\text{in},\beta}(x)\Big), & x < y, \end{cases} \tag{A1.5b}$$

where

$$V_{\alpha\beta} = S_0^{-1}\langle\alpha, \beta; \text{out}|0; \text{in}\rangle$$

and

$$\Lambda_{\alpha\beta} = S_0^{-1}\langle 0; \text{out}|\alpha, \beta; \text{in}\rangle$$

are amplitudes for pair creation and annihilation by an external field. Using these expressions we can obtain

$$(G * \bar{G})_\Sigma(x|z) = \mathrm{i}\sum_{\alpha,\beta} (f_{\text{out},\alpha}(x) V_{\alpha\beta} f_{\text{out},\beta}(z) + f_{\text{out},\alpha}(x)(V\bar{V})_{\alpha\beta}\bar{f}_{\text{out},\beta}(z))$$

$$= -\mathrm{i}\sum_{\alpha,\beta} (f_{\text{in},\alpha}(x)\bar{\Lambda}_{\alpha\beta} f_{\text{in},\beta}(z) + \bar{f}_{\text{in},\alpha}(x)(\Lambda\bar{\Lambda})_{\alpha\beta} f_{\text{in},\beta}(z)).$$

These relations show that the equality

$$(G * \bar{G})_\Sigma(x|z) = 0, \qquad x > \Sigma > z \tag{A1.6}$$

is a necessary and sufficient condition for the vanishing of matrices $\boldsymbol{V}$ and Λ and hence for the vacuum stability.

To obtain the equations connecting the Green function $G^{(\text{in})}(x|y)$ with $G(x|y)$ we note that

$$G^{(\text{in})}(x|y) = \mathrm{i}\begin{cases} \sum_\alpha f_{\text{in},\alpha}(x)\bar{f}_{\text{in},\alpha}(y), & x > y \\ \sum_\alpha f_{\text{in},\alpha}(y)\bar{f}_{\text{in},\alpha}(x), & x < y. \end{cases} \tag{A1.7}$$

Using equation (A1.5) one can obtain

$$(G * G^{(\text{in})})_\Sigma(x|z) = G(x|z), \qquad x > \Sigma > z, \tag{A1.8}$$

$$(\bar{G} * G^{(\text{in})})_\Sigma(x|z) = 0, \qquad x > \Sigma > z. \tag{A1.9}$$

These equations and a symmetry condition,

$$G^{(\text{in})}(x|y) = G^{(\text{in})}(y|x), \tag{A1.10}$$

define the Green function $G^{(\text{in})}(x|y)$ uniquely. To show this, we suppose that, for $x > y$, $G^{(\text{in})}(x|y)$ is written in the form

$$\begin{aligned} G^{(\text{in})}(x|y) = \mathrm{i}\sum_{\alpha,\beta} (&f_{\text{in},\alpha}(x)B_{\alpha\beta}f_{\text{in},\beta}(y) \\ &+ f_{\text{in},\alpha}(x)C_{\alpha\beta}\bar{f}_{\text{in},\beta}(y) + \bar{f}_{\text{in},\alpha}(x)D_{\alpha\beta}f_{\text{in},\beta}(y) \\ &+ \bar{f}_{\text{in},\alpha}(x)E_{\alpha\beta}\bar{f}_{\text{in},\beta}(y)). \end{aligned} \tag{A1.11}$$

Equation (A1.8) gives $\boldsymbol{B} = 0$, $\boldsymbol{C} = \boldsymbol{I}$. If we put expression (A1.11) into (A1.9), we obtain $\boldsymbol{D} = \boldsymbol{E} = 0$. Using (A1.10) we find that $G^{(\text{in})}(x|y)$ coincides with (A1.7).

Appendix 2. Proof of the vacuum stability condition

For a free causal Green function $G_0(x|y)$ in an empty Minkowski space the following relations are satisfied:

$$(G_0 * G_0)_\Sigma(x|z) = 0, \qquad x < \Sigma,\ z < \Sigma \quad \text{or} \quad x > \Sigma,\ z > \Sigma, \tag{A2.1}$$

$$(G_0 * \bar{G}_0)_\Sigma(x|z) = 0, \qquad x > \Sigma > z \quad \text{or} \quad z > \Sigma > x. \tag{A2.2}$$

Here Σ is a total Cauchy surface in a space–time without boundaries.

Denote now

$$Z^U_{F,K}(x|z) = \int_{y \in U} \mathrm{d}y F(x|y) \frac{\overleftrightarrow{\partial}}{\partial y^0} K(y|z), \tag{A2.3}$$

where $F(x|y)$ and $K(y|z)$ are two arbitrary functions and U is a part of the $y^0 = 0$ hyperplane. Pass to new variables ξ^μ,

$$y^\mu = (a^2/\xi^2)\xi^\mu, \tag{A2.4}$$

in which the equations $y^0 = 0$, $y \in U$ can be rewritten as

$$\xi^0 = 0, \qquad \xi \in JU, \tag{A2.5}$$

where JU is an image of the region U under the inversion transformation (A2.4). If we take into account that

$$\frac{\xi^2}{a^2} \frac{\partial}{\partial \xi^0}\bigg|_{\xi^0=0} = \frac{\partial}{\partial y^0}\bigg|_{y^0=0},$$

then it is not difficult to get

$$Z^U_{F,K}(x|z) = \int_{\xi \in JU} \mathrm{d}\xi [J_a(\xi) F(x|\xi)] \frac{\overleftrightarrow{\partial}}{\partial \xi^0} [J_a(\xi) K(\xi|z)], \tag{A2.6}$$

where

$$J_a(\xi) f(\xi) = -\frac{a^2}{\xi^2} f\left(\frac{a^2}{\xi^2}\xi\right).$$

Because the vacuum stability condition (A1.6) does not depend on the choice of a particular Cauchy surface Σ, we can take part of the $y^0 = 0$ hyperplane as a Σ surface.

In this case equation (A1.6) can be rewritten as

$$Z^U_{G,\bar{G}}(x|z) = 0, \qquad x^0 > 0 > z^0 \tag{A2.7}$$

For the exterior (interior) problem A, $y \in U$ when $|y| \geq a$ ($|y| \leq a$). For problem B, $y \in U$ when $a \leq |y| \leq b$. If the point y lies in the hyperplane $y^0 = 0$, then $y^2 \geq 0$ and we have for the Green functions in the problem under consideration

$$J(y) G(x|y) = G(x|y), \qquad J(y) G(y|z) = G(y|z).$$

If we remember now that

$$\bar{G}(y|z) = K(y) \bar{G}_0(y|z),$$

where

$$K(y) = I + J_a(y)$$

for problem A and

$$\boldsymbol{K}(y)=(\boldsymbol{I}-\boldsymbol{J}_a\boldsymbol{J}_b)^{-1}(\boldsymbol{J}_a+\boldsymbol{J}_a\boldsymbol{J}_b)+(\boldsymbol{I}-\boldsymbol{J}_b\boldsymbol{J}_a)^{-1}(\boldsymbol{J}_b+\boldsymbol{J}_b\boldsymbol{J}_a)$$

for problem B, then equation (A2.6) can be used to obtain the vacuum stability condition in the form

$$Z^{R^3}_{G,\bar{G}_0}(x|z)=\int_{y^0=0}\mathrm{d}y G(x|y)\frac{\overleftarrow{\partial}}{\partial y^0}\bar{G}_0(y|z)=0, \qquad x^0>0>z^0.$$

In the case of the exterior problem A and in problem B $x^2>0$and hence $G(x|y)=\boldsymbol{K}(x)G_0(x|y)$. Thus we have

$$(G*\bar{G})_\Sigma(x|z)=\boldsymbol{K}(x)\int_{y^0=0}\mathrm{d}y G_0(x|y)\frac{\overleftarrow{\partial}}{\partial y^0}G_0(y|z)=0, \qquad x^0>0>z^0.$$

The last equality is written because of property (A2.2) of the free causal Green function.

In the case of the interior problem A, if $x^2>0$, then the above consideration is also valid. If $x^2<0$, then

$$G(x|y)=G_0(x|y)+\boldsymbol{J}_a(x)\bar{G}_0(x|y),$$

and using properties (A2.1) and (A2.2) one can also verify the fulfilment of the vacuum stability condition.

References

Candelas P and Deutsch D 1977 *Proc. R. Soc.* A **354** 79
Chitre D M and Hartle J B 1977 *Phys. Rev.* D **16** 251
Davies P C W and Fulling S A 1977 *Proc. R. Soc.* A **356** 237
DeWitt B S 1975 *Phys. Rep.* C **19** 295
Fulling S A and Davies P C W 1976 *Proc. R. Soc.* A **348** 393
Hartle J B and Hawking S W 1976 *Phys. Rev.* D **13** 2188
Lapedes A S 1978 *Phys. Rev.* D **17** 2556
Menskii M B 1974 *Teor. Mat. Fiz.* **18** 190
Moore G T 1970 *J. Math. Phys.* **11** 2679
Rumpf H 1976 *Phys. Lett.* B **61** 272
Schwinger J 1970 *Particles, Sources and Fields* (Massachusetts: Addison-Wesley)